# Comparing Structured and Unstructured Models

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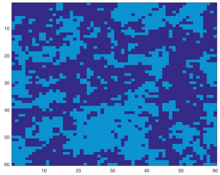
# A Story about Voting

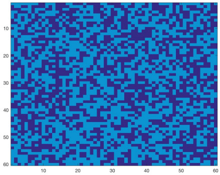
- Suppose there are households on a 2D grid and people living near each other tend to have have similar political opinions.
- In bipartisan voting, the outcome is 60:40 and the data about who voted for whom is available.
- Accusation of rigging : that someone assigned votes randomly to people to get this result.
- Can we decide if the voting data is fraudulent or legitimate?

(a) voting households

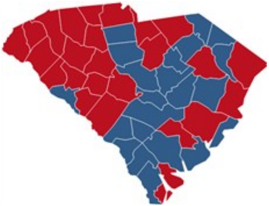
# A Story about Voting




(a) probably legitimate




(b) probably fraudulent



(c) South Carolina 2008 (courtesy: NBC)

# A Story about Voting : Key Features

- Only one sample available.
- Underlying graph structure is known. (Structured Model)
- Exact extent of interaction (peoples' influence on neighbors) unknown.
- Test against data coming from distribution without the said graph structure. (Unstructured Model)
- The data generated is intimately linked to the structure of the underlying network.
- We will use Ising model to for voter behavior.

# Probabilistic Models Involving Networks

- Data is generated by networks or network itself is data.
- Ising model : graphical model for binary data over networks.
- Erdős-Rényi graphs, Exponential Random Graphs, Stochastic Block Model etc. are used to model social networks.

# Ising Model

## Ising Model

Graph $G = (V, E)$. Random variable along each vertex $\in \{-1, 1\}$. Each node interacts only with the neighbors in the graph.

$$p(x) \propto \exp \sum_{(i,j)\in E} J_{ij} x_i x_j = \exp \frac{1}{2} x^\intercal J x$$

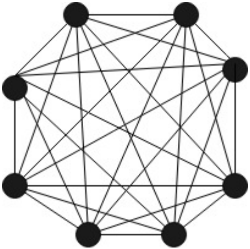
## Curie-Weiss Model

Ising model over complete graph. 'Inverse temperature' $\beta$,

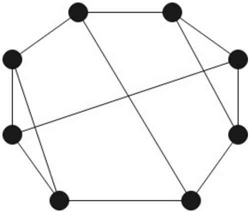
$$p(x) \propto \exp \frac{n\beta}{2} m^2$$

Where $m(x) := \frac{\sum_{i=1}^n x_i}{n}$

# Interaction Diagram



(a) Curie Weiss Interaction



(b) 3 Regular Graph Interaction

Figure: Interaction Diagram

# Vote Rigging Model

$x \in \{-1, 1\}^n$ : $-1$ is vote for first party, $+1$ is vote for second party.

**$\mathbf{H_0}$:Legitimate voting**

$$p(x) \propto \exp \beta(A(x) - D(x))$$

$$A(x) = \# \text{ agreements between neighbors}$$

$$D(x) = \# \text{ disagreements between neighbors}$$

$\beta$ unknown. d neighbors for everyone $\implies$ d regular Ising model.
Intuition: Promotes agreement between neighbors.

**$\mathbf{H_1}$:Fraudulent voting**

Curie-Weiss model or everyone votes independently with some partisan bias. Unknown inverse temperature or bias.

# Random Graph Models

## Erdős-Rényi Model

$G(n,p)$ : Edges are independently chosen with probability $p$.

$$\begin{aligned}\mu(G) &= p^{|E(G)|}(1-p)^{\binom{n}{2}-|E(G)|} \\ &\propto e^{\log\left(\frac{p}{1-p}\right)|E(G)|}\end{aligned}$$

## Exponential Random Graph Model

$$p(G) \propto \exp\left[\beta_1|E(G)| + \frac{\beta_2}{n}N_G(\bigvee) + \dots\right]$$

Where $\beta \in \mathbb{R} \times (\mathbb{R}^+)^{K-1}$. Intuition: Promotes the appearance of sub-graphs in addition to edges.

# Unstructured Models

- Curie-Weiss model has a lot of symmetries.
- Fraudulent voting data from Curie-Weiss model: it doesn't consider the 'community structure' where neighbors influence each other.
- $G(n, p)$ assigns the same probability to graphs with the same number of edges, irrespective of the structural properties.
- We concentrate on Ising model to illustrate our method. Similar methods can be used for ERGMs.

# Correlation Decay and High Temperature

- Interactions are small enough and no long range order.
- A node cannot influence nodes very far away.
- We can still distinguish structured and unstructured models in high temperature regime!
- Main result gives sharp thresholds above which it is possible to distinguish them and below which it is impossible to distinguish them.

# Hypothesis Testing Problem

We observe $x \in \{-1, 1\}^n$
$\mathbf{H_0}$ : Data from Curie-Weiss model at unknown temperature/ Independence model with unknown bias.
$\mathbf{H_1}$ : Data from $d$-regular Ising model, with known graph but unknown interaction strength $\beta$.

# Our Theorem: Ising Model

- Let $d = o(n)$. Statistical test to distinguish d-regular Ising model from Curie-Weiss/Independent voting model.
- Can distinguish with one sample with high probability when $\beta \gg \frac{1}{\sqrt{nd}}$, where $\beta$ interaction along each edge in the $d$ regular model.
- When $\beta \ll \frac{1}{\sqrt{nd}}$, there is no test to do this with constant number of independent samples: Our test is optimal!
- High temperature regime is $\beta \leq \Theta(\frac{1}{d})$. The statistical threshold lies deep inside it.

# An Intuitive View

- Let $A_m$ be the set of all states such that $\frac{\sum_i x_i}{n} = m$.
- Curie-Weiss model/ Independent voting model : $p(.|x \in A_m)$ uniform on $A_m$ because of symmetries in them.
- d regular Ising model: $q(x) \propto e^{\beta(A(x)-D(x))}$. Higher probability to states with higher number of agreements.
- We decide $H_1$ when $A(x) - D(x)$ 'is large enough'.
- This test is optimal!

# An Intuitive View

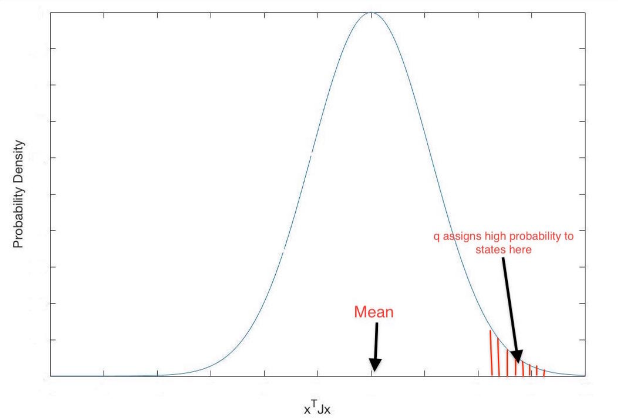



Figure: distribution of $A(x) - D(x)$ under uniform measure over $A_m$

# An Intuitive View

- To prove that this test works, we need to show that the right tail is large enough: Use central limit theorem ! (We apply Stein's method to show this)
- To prove the converse, we need exponential concentration inequalities

# Our Theorem: ERGM

- Consider ERGM$(\beta_1, \beta_2)$ with $K = 2$, with $H_2$ being the $V$ graph (  ).
- Same framework to distinguish ERGM$(\beta_1, \beta_2)$ from $G(n, p)$. Statistical test: Count the number of (  ) graphs!
- When $\beta_2 \gg \frac{1}{\sqrt{n}}$, our test can distinguish them using one sample with high probability.
- When $\beta_2 \ll \frac{1}{\sqrt{n}}$, there is no test to do this: Our test is optimal!
- Correlation decay regime threshold is $\beta \leq \Theta(1)$. The statistical threshold lies deep inside the correlation decay regime.

# Abstract Result : Conditions

We consider a sequence of probability spaces $(\Omega_n, \mathcal{F}_n, p_n)$, $n \in \mathbb{N}$.
Define measure $q$ as:

$$\frac{dq}{dp} = \frac{e^{\beta g}}{\mathbb{E}_p[e^{\beta g}]}$$

Let the following conditions hold.

- We can partition $\Omega = \cup_{m=1}^{M} A_m$ with disjoint sets $A_m$ such that $p(A_m) > 0 \ \forall \ i$.
- There exists $S_n \subset [M]$ such that $p_n(\cup_{m \in S_n} A_m) \geq 1 - \alpha_n$ for some sequence $\alpha_n \to 0$.
- Let $p^{(m)} = p(\cdot | x \in A_m)$. Let $X_m \sim p^{(m)}$. Let $e_m(g) := \mathbb{E}[g(X_m)]$ and $\sigma_m^2(g) := \mathrm{var}\,[g(X_m)]$. For all $m, m' \in S_n$,

$$0 < c \leq \frac{\sigma_m^2(g)}{\sigma_{m'}^2(g)} \leq C$$

# Abstract Result : Conditions

- $\sup_{m \in S_n} d_{\mathrm{KS}}\left(\mathcal{L}\left(\frac{g(X_m)-e_m(g)}{\sigma_m}\right), \mathcal{N}(0,1)\right) < \tau_n$ for some sequence $\tau_n \to 0$. We let $\sigma_n$ be any number such that $c\sigma_m \leq \sigma_n \leq C\sigma_m$ for some constants $c$ and $C$ for every $m \in S_n$.
- Let $\mathrm{var}(g(X)) = O(\sigma_n^2)$. Further, let

$$\log \mathbb{E}\left[e^{\beta(g(X)-\mathbb{E}g(X))}\right] \leq \frac{\beta^2 \nu}{1-|\beta| c}$$

$\forall\ |\beta| < \frac{1}{c}$ such that $c \leq A\sigma_n$ and $\nu \leq B\sigma_n^2$ for absolute constants $A, B$ independent of $n$.

# Canonical Statistical Test

Function $m(x)$ such that $m(x) = m_0$ iff $x \in A_{m_0}$.
$H_0 : X \sim p$
$H_1 : X \sim q$ for some $\beta_n$
We call the following statistical test to distinguish $H_0$ and $H_1$ the canonical test with parameter $T \geq 0$:

**Definition (Canonical Test)**

Given a sample $X$, we define the following decision function $\mathcal{D}(X) \in \{H_0, H_1\}$:

1. if $m(X) \notin S_n$ then $\mathcal{D}(X) = H_1$
2. if $m(X) \in S_n$ and $\frac{g(X) - e_{m(X)}(g)}{\sigma_{m(X)}} \geq T$ then $\mathcal{D}(X) = H_1$
3. otherwise $\mathcal{D}(X) = H_0$

The test doesn't depend on $\beta_n$

# Abstract Result

**Theorem (Main Result)**

*If the conditions above hold,*

$$\lim_{n\to\infty} d_{\mathrm{TV}}(p_n, q_n) = 1 \quad \text{if } \beta_n\sigma_n \to \infty \tag{1}$$

$$\lim_{n\to\infty} d_{\mathrm{TV}}(p_n, q_n) = 0 \quad \text{if } \beta_n\sigma_n \to 0 \tag{2}$$

*If $\beta_n\sigma_n \geq L_n$ for a known sequence $L_n \to \infty$, ($\beta_n$ being possibly unknown) the canonical test can distinguish between $p$ and $q$ with high probability with a single sample for a particular choice $T_n \to \infty$ depending only on $L_n$ and $\tau_n$. The probability of error can be bounded above by a function of $\alpha_n$, $T_n$ and $L_n$ tending to 0.*

# Application to Ising Models

- To compare $d$-regular Ising model over graph $G$ at temperature $\beta$ with Curie-Weiss model at temperature $\beta'$, take $p(\cdot)$ to be the Curie-Weiss model.
- $A_{m_0} := \{x : m(x) = m_0\}$ and $g(x) = x^\intercal Jx - \frac{\beta'}{\beta} nm^2$ where $J$ is the adjacency matrix of $G$.
- $\sigma_n = \Theta(\sqrt{nd})$ uniformly $\forall m \in [-1+\delta, 1-\delta] =: S_n$
- $\sup_{m \in S_n} d_{\mathrm{KS}} \left( \mathcal{L}\left( \frac{g(X_m) - e_m(g)}{\sigma_m} \right), \mathcal{N}(0,1) \right) < C \sqrt[4]{\frac{d}{n}}$

# Application to ERGM

- To compare ERGM$(\beta_1, \beta_2)$ with $G(n,p)$, take $p(\cdot) \sim G(n,p)$
- $A_{m_0} := \{G = (V,E) : |E| = m_0\}$
- $g(G) = \left( n \left( \frac{\beta_1 - \frac{1}{2}\log\frac{p}{1-p}}{\beta_2} \right) E(G) + N_2(G) \right)$.
- $\sigma_n = \Theta(n^{3/2})$ uniformly $\forall m \in \left[\frac{\delta}{2}N, N(1-\frac{\delta}{2})\right] =: S_n$ where $N = \binom{n}{2}$
- $\sup_{m \in S_n} d_{\mathrm{KS}} \left( \mathcal{L}\left( \frac{g(X_m) - e_m(g)}{\sigma_m} \right), \mathcal{N}(0,1) \right) < C \sqrt[4]{\frac{1}{n}}$

Proof of both central limit theorems rely on application of Stein's method using suitable exchangeable pairs.

Questions ?